Investigating Knotted Stellarator Fields using Hypersphere Coordinates

Kirstin Koepnick, Advisors: Dr. C.B. Smiet, B. Israeli

07 August 2020

SULI 2020

Introduction

Motivation

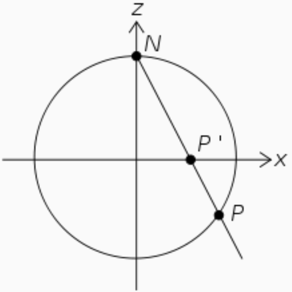
- What is the best container for fusion?
- What happens if this container was a knotted donut?

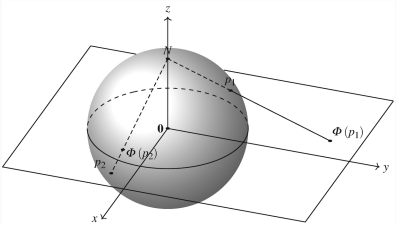
Project Goals

- Generate a wide class of divergence free vector fields using stereographically projected complex coordinates on the hypersphere.
- Kedia et al. PRL 117, no. 27 (2016): 274501

The Hypersphere and Stereographic Projections

- $S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}.$





The Specifics

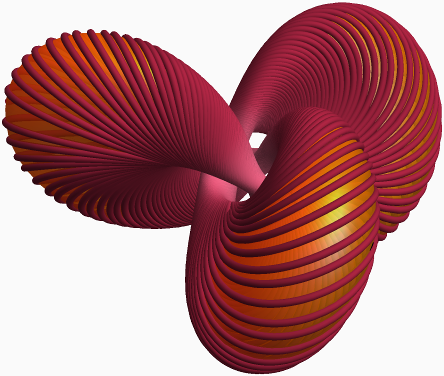
Kedia's Formalism

- Kedia *et al.* gives us a recipe to constructing knotted divergence free vector fields using complex coordinates as well as includes helicity derivations.
- They construct a family of null solutions to Maxwell's equations in free space, but we just take this recipe (without worrying about time dependence) and harness the mathematics in order to design a stellarator.

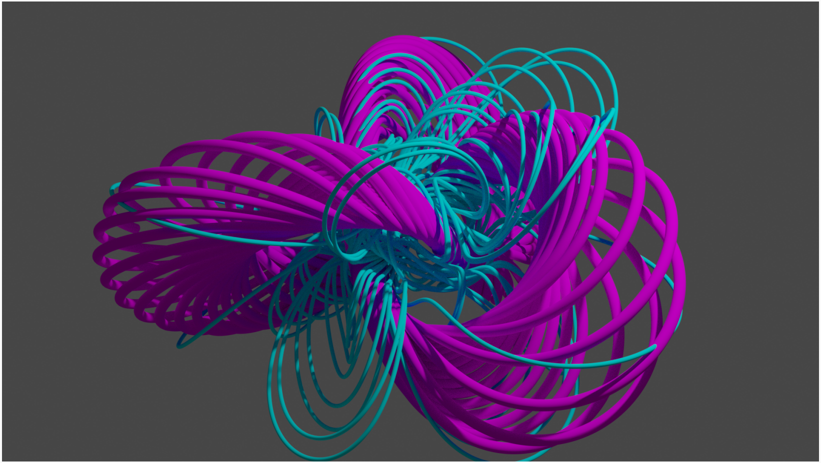
The Recipe

- $u = \frac{r^2-1+2iz}{r^2+1}$ and $v = \frac{2(x-iy)}{r^2+1}$ such that $|u|^2 + |v|^2 = 1$
- $F = Im[\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta]$ and $Re[\alpha\beta]$
  - $\alpha = u^p$ and $\beta = v^q$ for p, q coprime integers
  - Since $Im[\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta] \cdot \boldsymbol{\nabla} Re[\alpha\beta] = 0$, our field lines lie on constant surfaces of $Re[\alpha\beta]$.

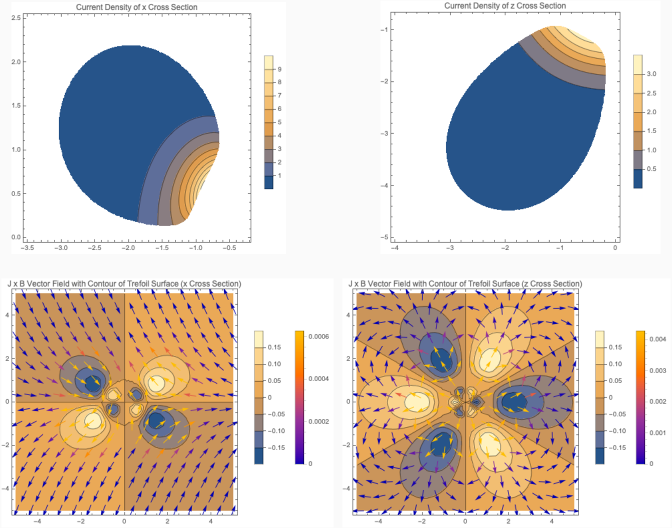
Our Fields



Current and Magnetic Fields



Current Density and $J \times B$



A special thanks to my advisors Chris Smiet and Ben Israeli.

This work was made possible by funding from the Department of Energy for the Summer Undergraduate Laboratory Internship (SULI) program. This work is supported by the US DOE Contract No. DE-AC02-09CH11466.

Investigating Knotted Stellarator Fields using Hypersphere Coordinates

K.E. Koepnick (Bates College), advised by C.B. Smiet, B.Y. Israeli (Princeton Plasma Physics Laboratory)

This work was made possible by funding from the Department of Energy for the Summer Undergraduate Laboratory Internship (SULI) program. This work is supported by the US DOE Contract No. DE-AC02-09CH11466

GOALS

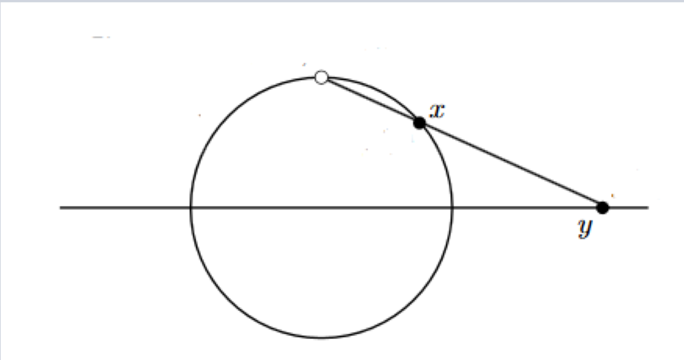
- **Generate a family of divergence free vector fields using stereographically projected points from the hypersphere, S^3**
- **Investigate these vector fields as potential knotted stellarator field configurations**

BACKGROUND: Hypersphere and Stereographic Projection

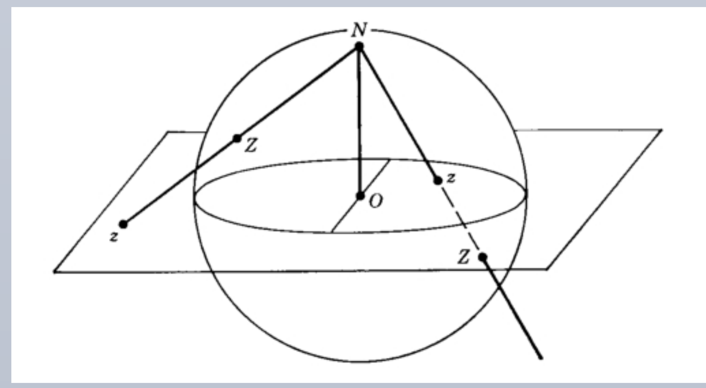
- An n-sphere is defined by a set of points from a radius at a set point. A 3-sphere is a sphere that exists in 4-dimensional space and is defined by:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

- Stereographic projection is a continuous map from an n-sphere to n-space. This map removes one point from an n-sphere (the "north pole" in most cases) and maps each point on the sphere to a corresponding point in n-space.
- To help understand: consider the unit circle and the stereographic projection defined by the picture below that sends the point x on the circle to the point y on the real line:



- We can also consider the 2-D case defined with the unit sphere and the real plane rather than real line:



- More concretely, we can define these stereographic projections generally as the map:

$$p : S^n / \{N\} \to \mathbb{R}^n$$

Generating Divergence Free Vector Fields

- This stereographic projection of u and v gives us the following functions:

$$u = \frac{r^2 - 1 + 2iz}{r^2 + 1} \text{ and } v = \frac{2(x - iy)}{r^2 + 1}$$
$$\text{such that } |u|^2 + |v|^2 = 1$$

- A divergence-free vector field is then given by the following expression:

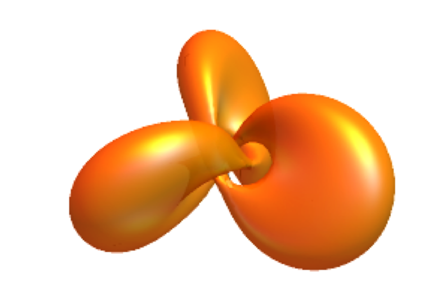
$$\mathbf{B} = Im[\nabla\alpha \times \nabla\beta]$$
$$\text{where } \alpha = u^p,\ \beta = v^q \text{ for } p, q \in \mathbb{Z}$$

- These field are analytic on R^3 and have a very interesting topology. Because the following identity holds:

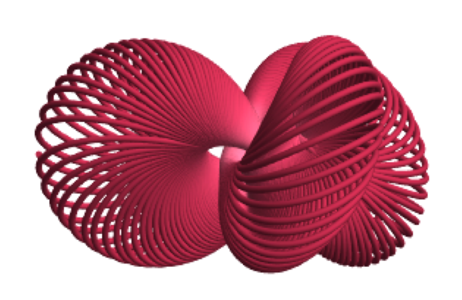
$$Im[\nabla\alpha \times \nabla\beta] \cdot \nabla Re[\alpha\beta] = 0$$

- These fields are isosurfaces on the following function. These fields lie on nested toroidal surfaces and the topology is defined by p and q integers.

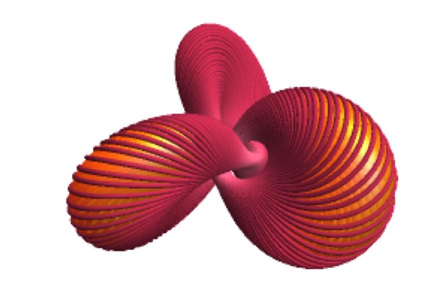
$$Re[\alpha\beta]$$



Trefoil Surface Defined by p=2, q=3



B-Field Lines

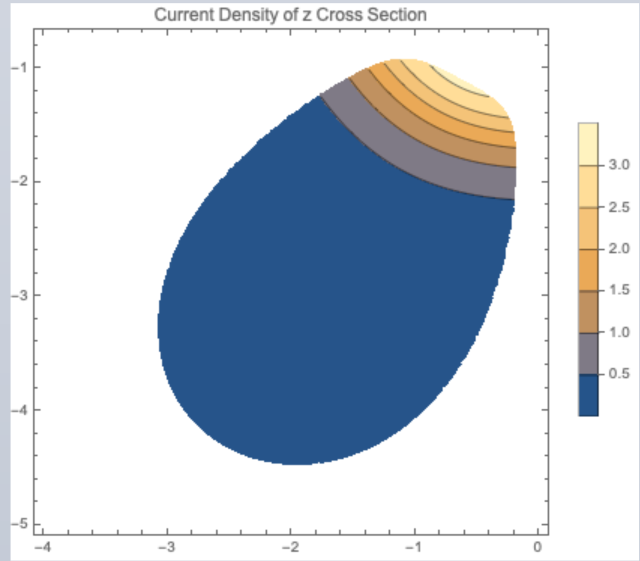


Surface and Field Lines

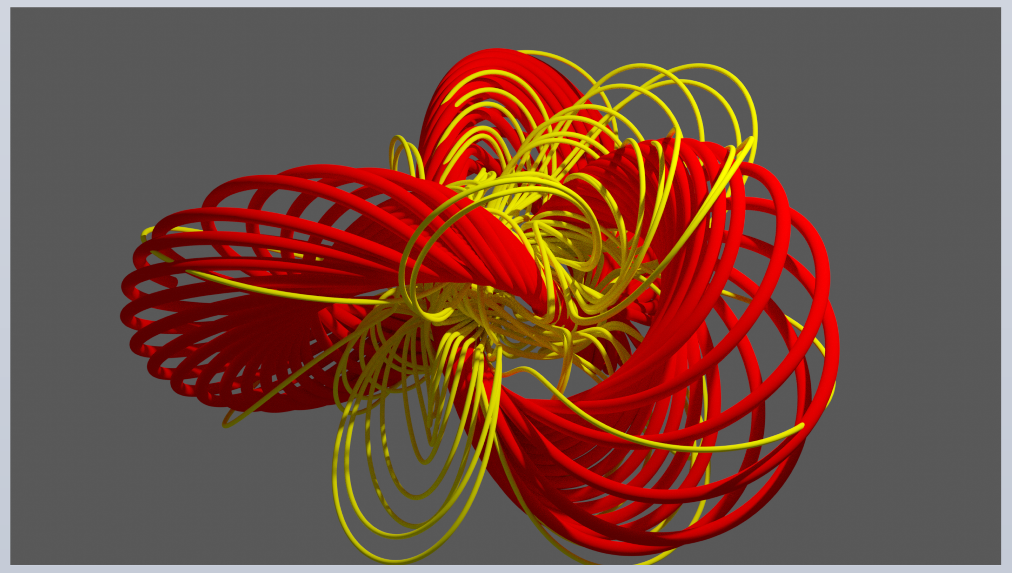
Current and Forces in these Configurations

- These fields are generated by a non-zero special current (J = curl of B) so they are not inherently current free. The current is not evenly distributed and parallel to magnetic field and are not in equilibrium (see first row of figures below). Topology conserving relaxation can be employed to generate fields with lower current (future work). The bottom row of figures refers to plots of the Lorentz force and direction in which magnetic forces would move to lower the energy.

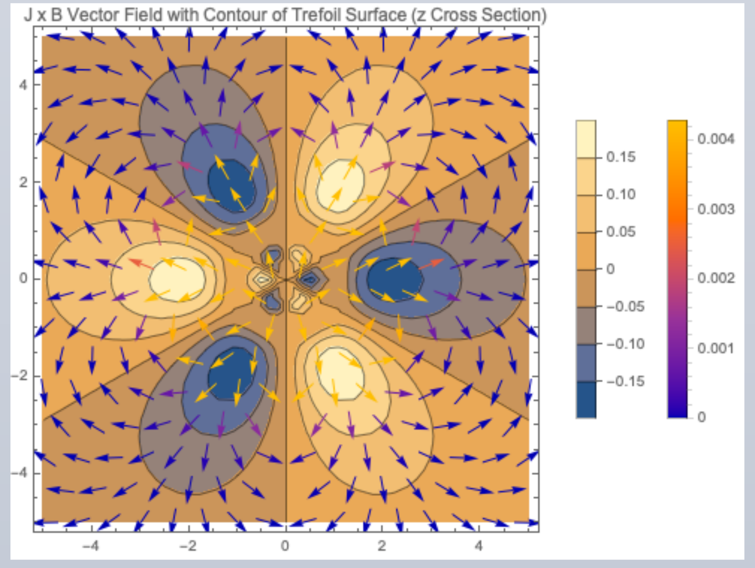
Contour Plot of Current Density in a Lobe in the z=0 Plane



Magnetic and Current Field Lines
Red = Magnetic Field, Yellow = Current



J x B Vector Field with Contour of Our Trefoil Surface in the z=0 Plane

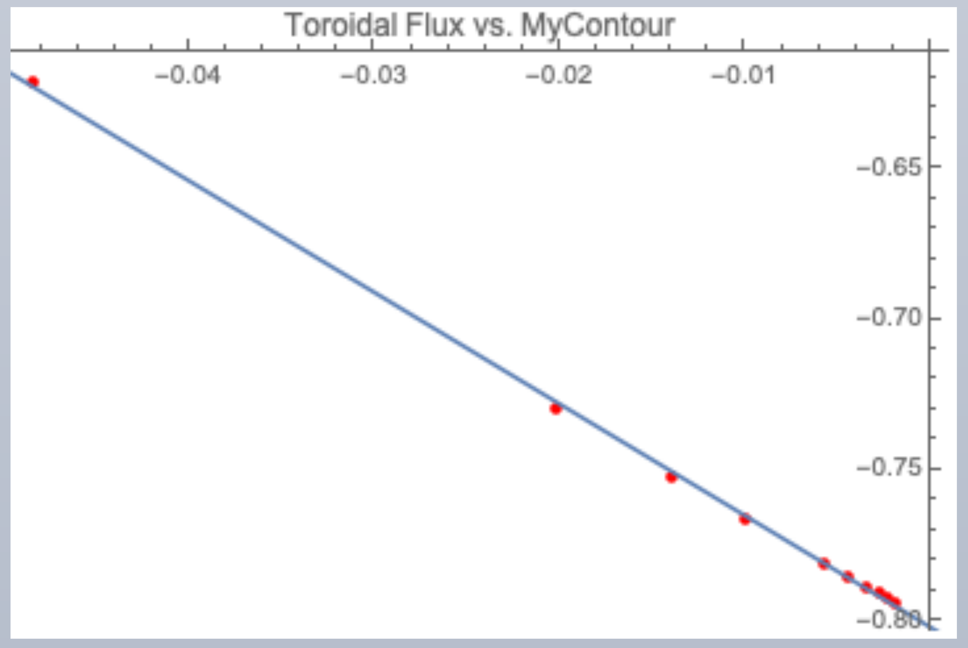


Toroidal Flux

- We also investigated the relationship between toroidal flux and the constant value (mycontour) to which Re[αβ] was set. This relationship was given by the following equation:

$$y = -0.802312 - 3.70443x$$

- Where y represents toroidal flux and x represents the "mycontour" value. As we can see, this relationship is very linear.

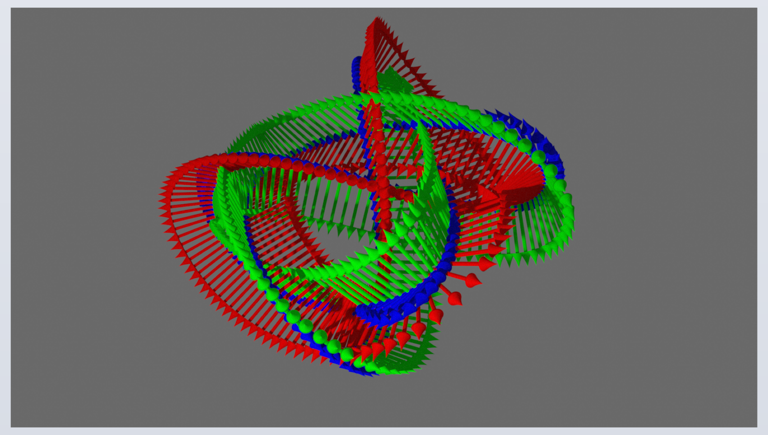


Helicity and Rotational Transform

- We know the global helicity to be given by $\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} d^3x$
- Where A is our vector potential:

$$\mathbf{A} = \frac{1}{2}(Im\{\alpha\nabla\beta\} - Im\{\beta\nabla\alpha\})$$

- We develop code to calculate the rotational transform in these knotted configurations. We calculating a moving Frenet-Serret frame for the magnetic axis and calculate the winding number as the number of rotation of the plane plus the number of times the field rotates in the frame itself.



CONCLUSIONS

- **Using complex differentiable functions composed of stereographically projected coordinates from the hypersphere allows us to neatly create knotted or twisted vector fields.**
- **Our fields still carry current in interesting ways.**
- **The rotational transform of our fields can be calculated using Frenet Serret frames done in python code.**
- **Future work: further investigation of the JxB force, rotational transform tunablity, and current minimization.**

REFERENCES

Papers:

[1] H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas, and W. T. M. Irvine, "Tying knots in light fields," Phys. Rev. Lett. **111**, 150404 (2013).
[2] H. Kedia, D. Foster, M. R. Dennis, and W. T. M. Irvine, "Weaving knotted vector fields with tunable helicity," Phys. Rev. Lett. **117**, 274501 (2016).
[3] A. Thompson, J. Swearngin, A. Wickes, and D. Bouwmeester, "Constructing a class of topological solitons in magnetohydrodynamics," Phys. Rev. E **89**, 043104 (2014).
[4] J. Hale, H. Buttanri, and H. Kocak, *Dynamics and Bifurcations*, Texts in Applied Mathematics (Springer New York, 1996).

CONTACT

- Kirstin Koepnick: kkoepnic@bates.edu
- Chris Smiet: csmiet@pppl.gov
- Ben Israeli: bisraeli@pppl.gov